Chapter 1 - Notation and Review of Newton's Laws (Part 1)

A. Space and Vectors:
   - Unit vectors, vector components and Cartesian orthonormal basis
   - Dot product and cross product

B. Time derivatives
   - Dot notation
   - Vector expressions

List of Examples and Proofs

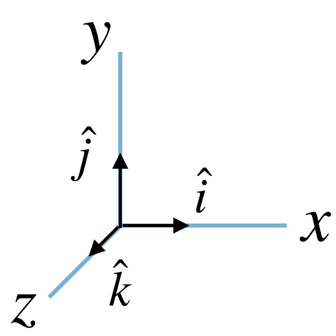
A. Space and Vectors

Unit Vectors and Orthonormal Basis

Unit vector = vector with unit length

Orthonormal Basis = Set of unit vectors that are mutually perpendicular and span the given space

Cartesian Orthonormal Basis defines (x,y,z) Cartesian Coordinate system:

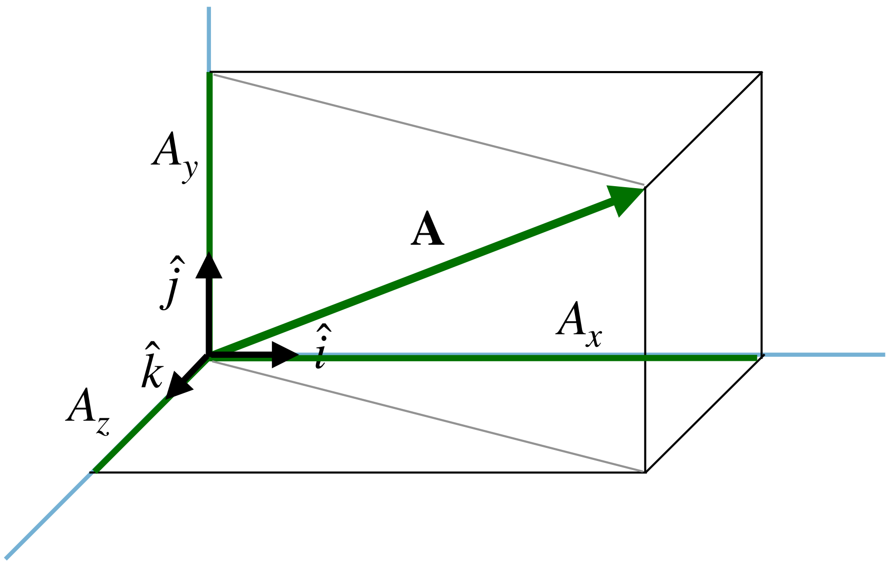


$\hat{i}$ points along x axis
$\hat{j}$ points along y axis
$\hat{k}$ points along z axis

Unit vectors wear a "hat": $\hat{i}, \hat{j}, \hat{k}$

Vector Components

The **vector components** of a vector $\mathbf{A}$ along the $\hat{i}$, $\hat{j}$, $\hat{k}$ axes are defined as A_x, A_y and A_z:



$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

The **position vector** $\mathbf{r}$ is a special case, since its components are often just written as $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ rather than $\mathbf{r} = r_x\hat{\mathbf{i}} + r_y\hat{\mathbf{j}} + r_z\hat{\mathbf{k}}$

Alternate Unit Vector Notation

Instead of writing the unit vectors as $\hat{i}$, $\hat{j}$, $\hat{k}$, sometimes it is convenient to write them as $\hat{x}$, $\hat{y}$, $\hat{z}$ or $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$. These notations are equivalent.

$$\begin{matrix} \hat{i} & & \hat{x} & & \hat{e}_1 \\ \hat{j} & \Leftrightarrow & \hat{y} & \Leftrightarrow & \hat{e}_2 \\ \hat{k} & & \hat{z} & & \hat{e}_3 \end{matrix}$$

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

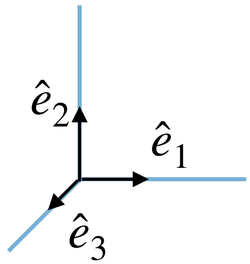
For example, the following are equivalent ways of writing the same vector:

$$\mathbf{r} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

$$\mathbf{r} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - 3\hat{\mathbf{z}}$$

$$\mathbf{r} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - 3\hat{\mathbf{e}}_3$$

$\hat{\mathbf{e}}_i$ notation



$$\mathbf{r} = r_1\hat{\mathbf{e}}_1 + r_2\hat{\mathbf{e}}_2 + r_3\hat{\mathbf{e}}_3$$

Advantage of the $\hat{\mathbf{e}}_i$ notation is that **summation notation** can be used. For example, the vector $\mathbf{r}$ can be written as a sum of the products of the vector components r_i with the unit vectors $\hat{\mathbf{e}}_i$

$$\mathbf{r} = \sum_{i=1}^{3} r_i\hat{\mathbf{e}}_i$$

Question

Each of the following equations contains a notational error. Find it.

$$A = A_x \hat{x} + A_y \hat{y}$$

$$A = \overrightarrow{A}_x \hat{x} + \overrightarrow{A}_y \hat{y}$$

$$\mathbf{B} = B_x \vec{x} + B_y \vec{y}$$

$$C = \overrightarrow{A} + \overrightarrow{B}$$

Vector Notation

Magnitude of vector:

$$|\mathbf{r}| = r = \sqrt{r_x^2 + r_y^2 + r_z^2} = \sqrt{\sum_{i=1}^{3} r_i^2}$$

Notation Summary:

Note: vector symbols must be either **bolded** or have an arrow over them

Vector position: $\mathbf{r}$ or $\vec{r}$

Unit vector (wears a hat): $\hat{\mathbf{r}}$

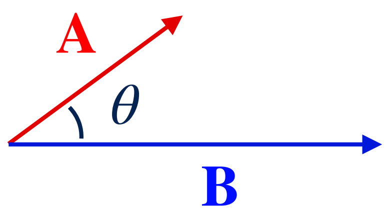
Vector magnitude (scalar): $|\mathbf{r}|$ or r

Dot Product

Given 2 vectors: $\mathbf{A} = \sum_{i=1}^{3} A_i \hat{\mathbf{e}}_i$ and $\mathbf{B} = \sum_{i=1}^{3} B_i \hat{\mathbf{e}}_i$

The dot product or "inner" product or "scalar" product is:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= \sum_{i=1}^{3} A_i B_i$$



$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

Dot Product of Orthonormal Unit Vectors

The following relationship defines a system of orthonormal unit vectors $\hat{e}_i$ and $\hat{e}_j$:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

where δ_{ij} is the **Kronecker delta** symbol:

$$\delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$$

When a unit vector is "dotted" with itself, we get 1. For example, $\hat{e}_1 \cdot \hat{e}_1 = 1$. When it is dotted with a different (perpendicular) unit vector we get zero. For example, $\hat{e}_1 \cdot \hat{e}_2 = 0$.

$\hat{e}_i$ \ $\hat{e}_j$	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$
$\hat{e}_1$	1	0	0
$\hat{e}_2$	0	1	0
$\hat{e}_3$	0	0	1

Properties of the Kronecker Delta Symbol

The Kronecker delta symbol is: $\delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$

It is easy to show the following:

$$\sum_{j=1}^{N} \left(\delta_{ij} A_j \right) = A_i$$

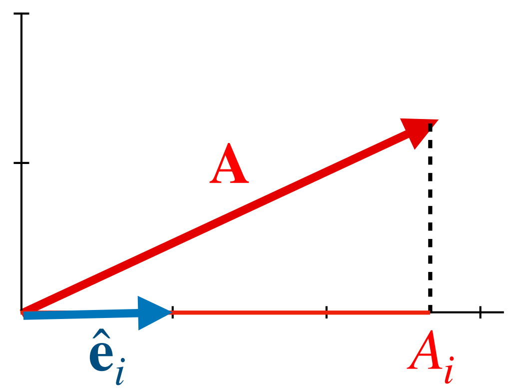
Notice: the A_j inside the sum switches to an A_i on r.h.s. of the equation

This is true because all the δ_{ij} values with $i \neq j$ are zero, so the only contribution to the sum is when $j = i$.

Use Dot Product to Extract Vector Components

Given a set of unit vectors $\hat{\mathbf{e}}_i$ defining a coordinate system, the vector components A_i along each coordinate axis are given by

$$A_i = \mathbf{A} \cdot \hat{\mathbf{e}}_i$$



Try to use the properties of the Kronecker delta symbol to prove this result. The solution is on the next slide.

Proof that $A_i = \mathbf{A} \cdot \hat{\mathbf{e}}_i$

Consider the following: $\mathbf{A} \cdot \hat{\mathbf{e}}_i$

Expand the vector A on the $\hat{\mathbf{e}}_j$ basis: $= \left(\sum_{j=1}^{3} A_j \hat{\mathbf{e}}_j \right) \cdot \hat{\mathbf{e}}_i$

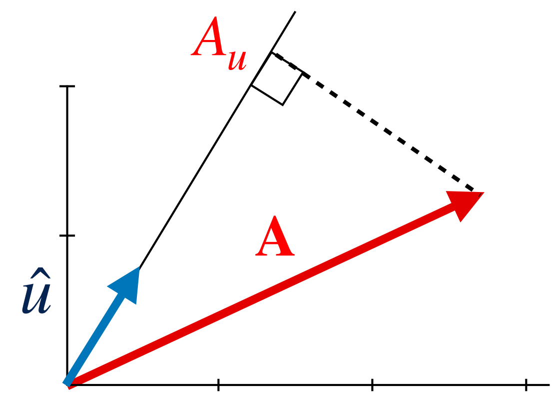
Bring $\hat{\mathbf{e}}_i$ into the sum: $= \sum_{j=1}^{3} \left(A_j \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i \right)$

Substitute $\hat{e}_i \cdot \hat{e}_i = \delta_{ij}$ $= \sum_{j=1}^{3} A_j \delta_{ij}$

From property of Kronecker deltas: $= A_i$ ✓

More Dot Product Properties

Given **any** unit vector $\hat{u}$ and vector $\mathbf{A}$, the vector component A_u in the direction defined by $\hat{u}$ is given by $A_u = \mathbf{A} \cdot \hat{\mathbf{u}}$

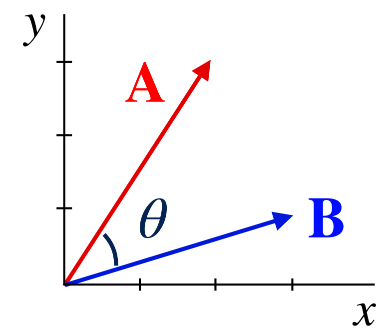


The dot product of any vector with itself is the square of its magnitude:

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

If two vectors $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{A} \cdot \mathbf{B} = 0$, then the the vectors are perpendicular (i.e. orthogonal).

Dot Product **Example**



Find the angle between the following two vectors:

$\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\mathbf{B} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$

Solution:

We'll use the result that $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$.

Calculate the dot product: $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y$

$$= (2)(3) + (3)(1) = 9$$

Next, find the magnitudes of each vector: $A = \sqrt{A_x^2 + A_y^2} = \sqrt{13}$

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{10}$$

Finally, solve for the angle θ

$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{9}{\sqrt{13}\sqrt{10}}\right) = 38^\circ$$ ✓

Cross Product

Given 2 vectors: $\mathbf{A} = \sum_{i=1}^{3} A_i \hat{\mathbf{e}}_i$ and $\mathbf{B} = \sum_{i=1}^{3} B_i \hat{\mathbf{e}}_i$

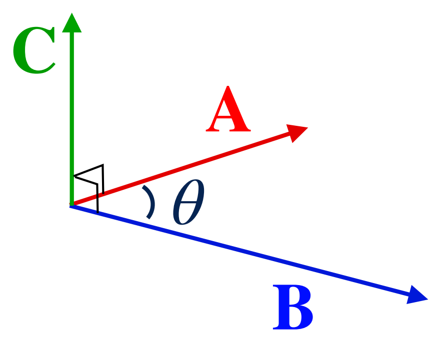
The cross product is $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ where the vector components are

$$C_x = A_y B_z - A_z B_y$$
$$C_y = A_z B_x - A_x B_z$$
$$C_z = A_x B_y - A_y B_x$$

or, equivalently

$$\mathbf{C} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

The cross product is a vector that is always perpendicular to the plane defined by vectors $\mathbf{A}$ and $\mathbf{B}$



The **magnitude** of the cross product $\mathbf{A}$ and $\mathbf{B}$ is

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

Cross Product

The determinant of the 3×3 matrix can be written as the sum of three 2×2 determinants like this:

$$\mathbf{C} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

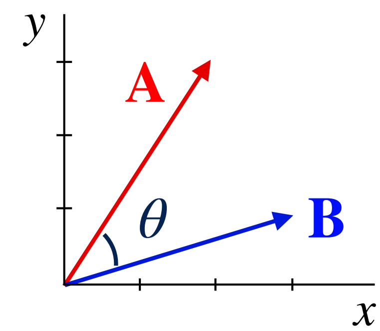
$$= \hat{\mathbf{i}} \det \begin{bmatrix} A_y & A_z \\ B_y & B_z \end{bmatrix} - \hat{\mathbf{j}} \det \begin{bmatrix} A_x & A_z \\ B_x & B_z \end{bmatrix} + \hat{\mathbf{k}} \det \begin{bmatrix} A_x & A_y \\ B_x & B_y \end{bmatrix}$$

$$= \hat{\mathbf{i}}(A_y B_z - A_z B_y) - \hat{\mathbf{j}}(A_x B_z - A_z B_x) + \hat{\mathbf{k}}(A_x B_y - A_y B_x)$$

This expression is equivalent to the result on the previous slide

Cross Product Example

Find the cross product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ of the following two vectors: $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\mathbf{B} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$



Solution:

$$\mathbf{C} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = \hat{\mathbf{i}}(A_yB_z - A_zB_y) - \hat{\mathbf{j}}(A_xB_z - A_zB_x) + \hat{\mathbf{k}}(A_xB_y - A_yB_x)$$

$$\mathbf{C} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \hat{\mathbf{i}}(3 \cdot 1 - 0 \cdot 1) - \hat{\mathbf{j}}(2 \cdot 1 - 0 \cdot 1) + \hat{\mathbf{k}}(2 \cdot 1 - 3 \cdot 1)$$

$$= 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}} \quad \checkmark$$

Dot Product and Cross Product Properties

$\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are vectors. The following are true:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$ for any $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \times \mathbf{A} = 0$$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$$

B. Time Derivatives

Time Derivatives

We use the **dot notation** for time derivatives. The number of dots represents the number of derivatives

Examples:

velocity (1st derivative): $\dot{x} = \dfrac{dx}{dt} = v$

acceleration (2nd derivative): $\ddot{x} = \dfrac{d^2x}{dt^2} = a$

jerk (3nd derivative): $\dddot{x} = \dfrac{d^3x}{dt^3}$

Differentiating a Vector in Cartesian Coordinates

Because Cartesian unit vectors have fixed directions, we can treat them as constants when differentiating a vector:

Example: Find the vector velocity by differentiating the position vector.

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right]$$

We write $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$, etc

$$\dot{\mathbf{r}} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}}$$

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$$

We now define $v_x = \dot{x}$, $v_y = \dot{y}$, etc

$$\mathbf{v} = \dot{\mathbf{r}}$$

Time Derivatives

The product rule for differentiation applies to dot products and cross products:

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \dot{\mathbf{A}}\cdot\mathbf{B} + \mathbf{A}\cdot\dot{\mathbf{B}}$$

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \dot{\mathbf{A}}\times\mathbf{B} + \mathbf{A}\times\dot{\mathbf{B}}$$

Warning!! Don't overlook the dots above the vectors. They can be easy to miss.

Show: $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \dot{\mathbf{A}}\cdot\mathbf{B} + \mathbf{A}\cdot\dot{\mathbf{B}}$

Solution:

Write the dot product as sum

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d}{dt}\sum A_i B_i$$

$$= \sum \left(\dot{A}_i B_i + A_i \dot{B}_i\right)$$

$$= \sum \dot{A}_i B_i + \sum A_i \dot{B}_i$$

$$= \dot{\mathbf{A}}\cdot\mathbf{B} + \mathbf{A}\cdot\dot{\mathbf{B}}$$ ✅

Vector Time Derivative Example

Find the time derivative $\frac{d}{dt}|\mathbf{r}|^2$

Solution:

$$\frac{d}{dt}|\mathbf{r}|^2 = \frac{d}{dt}\mathbf{r}\cdot\mathbf{r}$$

$$= \dot{\mathbf{r}}\cdot\mathbf{r} + \mathbf{r}\cdot\dot{\mathbf{r}}$$

$$= 2\dot{\mathbf{r}}\cdot\mathbf{r}$$

$$= 2\mathbf{v}\cdot\mathbf{r}$$

If the velocity vector is perpendicular to the position vector (as in uniform circular motion), then the derivative is zero.